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ON THE ONE-SIDED WIENER'S THEOREM FOR THE MOTION GROUP ON $\mathbf{R}^N$

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ABSTRACT

It is proved that the one-sided Wiener's Theorem does not hold for the motion group $SO(N) \rtimes \mathbf{R}^N$. That is, there exists a proper closed right ideal in $L_1(SO(N) \rtimes \mathbf{R}^N)$ which is not contained in any closed maximal right ideal.

1. Introduction and preliminaries

Wiener's classical result states that every proper closed ideal in $L_1(\mathbf{R})$ is contained in a regular maximal ideal. Segal and Godement generalized Wiener's theorem to arbitrary abelian locally compact groups.

The two-sided analogue of Wiener's theorem fails to hold for $SL(2,\mathbf{R})$ [1] and for every noncompact connected semisimple Lie group [3]. However, Wiener's theorem was generalized by H. Leptin to all connected nilpotent Lie groups and to all semi-direct products of abelian groups [3]. This result was extended to all locally compact motion groups [2].

In [8] we proved that the one-sided analogue of Wiener's theorem already fails to hold for the motion group of the plane. In this paper we generalize this result to the motion group of $\mathbf{R}^N$ obtaining the following: there exists a proper closed right ideal in $L_1(SO(N) \rtimes \mathbf{R}^N)$ which is not contained in any closed maximal right ideal.

In section 2 we discuss the structure of right-translation invariant subspaces of $L_\infty(SO(N) \rtimes \mathbf{R}^N)$ for general N which is more complicated than the case $N = 2$.

In section 3 Wiener's theorem is reduced to a radial spectral analysis problem in $L_\infty(\mathbf{R}^N)$.

In section 4 we introduce some results due to Varopoulos [6] concerning pseudo-measures supported on spheres, and obtain estimates for their Fourier coefficients.

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The proof of our main result is given in section 5.

I would like to express my thanks for many conversations with S. Sidney concerning this work.

We will need the following definitions and notation. The motion group G is the semi-direct product $G = \mathrm{SO}(N) \rtimes \mathbf{R}^N$ where $\mathrm{SO}(N)$ is the orthogonal group. The multiplication law in G is given by

$$(\sigma, x)(\tau, w) = (\sigma\tau, x + \sigma w), \qquad \sigma, \tau \in \mathrm{SO}(N), \quad x, w \in \mathbf{R}^N.$$

For $f \in L_\infty(G)$ let $M(f)$ denote the w*-closed right invariant subspace generated by f. Let S_R denote the sphere $\{x \in \mathbf{R}^N : \|x\| = R\}$ and let μ_R denote the normalized Lebesgue measure of S_R. For $f \in L_\infty(\mathbf{R}^N)$ let $\mathrm{Sp}(f)$ denote the spectrum of f. Let $\Gamma(\Gamma_0)$ denote the set of all matricial coefficients of the irreducible unitary (spherical) representations of $\mathrm{SO}(N)$ [7], and let $\hat{f}$ denote the Fourier transform of f.

Finally, for the theory of spherical harmonics we refer the reader to [5,7].

2. Right invariant subspaces

The right-invariant, w*-closed subspaces of $L_\infty(G)$ are characterized by the following [see 4]:

THEOREM 2.1. *Every right-invariant, w*-closed, non-trivial subspace of $L_\infty(G)$ contains a function g of the form*

$$g(\sigma, x) = \sum_i m_{i,j} \otimes f_i(\sigma, x)$$

where $m_{i,j} \in \Gamma$, $f_i \in L_\infty(\mathbf{R}^N)$ for some fixed j.

$M(g)$ is minimal only if $\mathrm{Sp}(f_i) \subseteq S_R$ for some $R > 0$.

PROOF. Let $f \in L_\infty(G)$, $f \neq 0$. Then $M(f)$ contains all functions φ such that

$$\varphi(\sigma, x) = f(\sigma\sigma', x + \sigma w) \qquad \text{for all } \sigma' \in \mathrm{SO}(N) \quad \text{and} \quad w \in \mathbf{R}^N.$$

By the Peter–Weyl theorem, there exists a matrix coefficient m_{i_0, j_0} of an irreducible unitary representation π of $\mathrm{SO}(N)$, such that the function g defined as

$$g(\sigma, x) = \int_{\mathrm{SO}(N)} f(\sigma\sigma', x)\overline{m_{i_0, j_0}}(\sigma')d\sigma' = \sum_{i=1}^{r} g_i \otimes m_{i, j_0}(\sigma, x)$$

is non-zero and belongs to $M(f)$, where

$$g_i(x) = \int_{\mathrm{SO}(N)} f(\sigma', x)\overline{m_{i, j_0}}(\sigma')d\sigma', \qquad i = 1, 2, \ldots, r$$

and r is the rank of π. To prove the last statement we argue as in the case $N = 2$ [8].

For $N = 2$ we showed [8] that $M(f)$ contains a function $g(\sigma, x) = \varphi(x)$ which enables us to reduce the case to a problem in $L_\infty(\mathbf{R}^2)$. However, this result is false for $N > 2$. For $N > 3$ this is mainly due to the fact that the set of irreducible representations of $\mathrm{SO}(N)$ provided by the spherical harmonics is incomplete. Unfortunately, even for $N = 3$ the situation is more complicated as described in:

THEOREM 2.2. *Let $\psi(\sigma, x) = \Sigma_i m_{i,j} \otimes f_i(\sigma, x)$ for some fixed j where $f_i \in L_\infty(\mathbf{R}^N)$, $\mathrm{Sp}(f_i)$ is bounded for $i = 1, 2, \ldots, 2l+1$, and $m_{i,j}$ are the matricial coefficients of the irreducible representation of $\mathrm{SO}(N)$ provided by the solid spherical harmonics of order l, H_i, $i = 1, 2, \ldots, 2l+1$. Then $M(\psi)$ contains a function $g(\sigma, x) = \varphi(x)$, $\varphi \in L_\infty(\mathbf{R}^N)$, $\varphi \neq 0$, if and only if*

$$\sum_{i=1}^{2l+1} \hat{f}_i \bar{H}_i \neq 0.$$

PROOF. Suppose that $\Sigma_{i=1}^{2l+1} \hat{f}_i \bar{H}_i \neq 0$. Let h be a radial function in $\varepsilon(\mathbf{R}^N)$ with $h(\lambda) \neq 0$ for some $\lambda \in \mathrm{Supp}(\Sigma \hat{f}_i \bar{H}_i)$. Then

$$\sum_{i=1}^{2l+1} f_i * \tilde{h}\bar{H}_i \neq 0, \qquad \text{where } \tilde{h}\bar{H}_i = \widehat{h\bar{H}_i}, \quad i = 1, 2, \ldots, 2l+1. \tag{2.1}$$

The subspace $M(\psi)$ contains the function ψ^* defined by

$$\begin{aligned}
\psi^*(\sigma, x) &= \int_{\mathbf{R}^N} \psi(\sigma, x - \sigma w)\tilde{h}(|w|)\bar{H}_j\left(\frac{w}{|w|}\right) dw \\
&= \sum_{i=1}^{2l+1} m_{i,j}(\sigma) \int_{\mathbf{R}^N} f_i(x - \sigma w)\tilde{h}(|w|)\bar{H}_j\left(\frac{w}{|w|}\right) dw \\
&= \sum_{k=1}^{2l+1} m_{k,j}(\sigma) \sum_{i=1}^{2l+1} \overline{m_{i,j}}(\sigma) \int_{\mathbf{R}^N} f_k(x - \xi)\tilde{h}(|\xi|)\bar{H}_i\left(\frac{\xi}{|\xi|}\right) d\xi.
\end{aligned}$$

Hence, $g \in M(\psi)$ where

$$g(\sigma, x) = \varphi(x) = \int_{\mathrm{SO}(N)} \psi^*(\sigma\sigma', x)\, d\sigma' = \sum_{i=1}^{2l+1} f_i * \tilde{h}\bar{H}_i \neq 0$$

which proves the "if" part.

Suppose now that $\Sigma_{i=1}^{2l+1} \hat{f}_i \bar{H}_i = 0$ and that $g(\sigma, x) = \varphi(x)$ belongs to $M(\psi)$. Let P_τ be a net in $L_1(G)$ such that

$$\psi_\tau \xrightarrow[\tau]{w^*} g$$

where

$$\begin{aligned}
\psi_\tau(\sigma, x) &= \int_G \psi(\sigma\sigma', x - \sigma w) P_\tau(\sigma', w) d\sigma' dw \\
&= \sum_{i=1}^{2l+1} \int_G m_{i,j}(\sigma\sigma') f_i(x - \sigma w) P_\tau(\sigma', w) d\sigma' dw \\
&= \sum_{i=1}^{2l+1} \int_G \sum_{k=1}^{2l+1} m_{i,k}(\sigma) m_{k,j}(\sigma') f_i(x - \sigma w) P_\tau(\sigma', w) d\sigma' dw \\
&= \sum_{i=1}^{2l+1} \sum_{k=1}^{2l+1} m_{i,k}(\sigma) \int_{\mathbf{R}^N} f_i(x - \sigma w) \tilde{P}_{\tau,k}(w) dw
\end{aligned} \tag{2.2}$$

and

$$\tilde{P}_{\tau,k}(w) = \int_{SO(N)} P_\tau(\sigma', w) m_{k,j}(\sigma') d\sigma'.$$

Let

$$C_{\tau,k,i}(w) = \int_{S_1} \overline{\tilde{P}_{\tau,k}}(|w| w') \bar{H}_i(w') dw'$$

be the "Fourier coefficients" of $\overline{\tilde{P}_{\tau,k}}$ with respect to the basis $\{H_i\}$, $i = 1,2,\ldots,2l+1$, and let

$$D_{\tau,k}(w) = \sum_{i=1}^{2l+1} C_{\tau,k,i}(|w|) H_i\left(\frac{w}{|w|}\right).$$

Then

$$D_{\tau,k}(\sigma^{-1} w) = \sum_{i=1}^{2l+1} \overline{C_{\tau,k,i}}(|w|) \left[\sum_{s=1}^{2l+1} \overline{m_{s,i}}(\sigma) \bar{H}_s\left(\frac{w}{|w|}\right)\right]. \tag{2.3}$$

It follows by (2.1) that

$$\chi_\tau(x) = \int_{SO(N)} \psi_\tau(\sigma, x) d\sigma \xrightarrow[\tau]{w^*} g(x).$$

One notices that by Peter–Weyl orthogonality relations we may replace $\tilde{P}_{\tau,k}$ in (2.2) by $D_{\tau,k}$. It follows, by (2.3), that for each τ

$$\chi_\tau(x) = \sum_{k=1}^{2l+1} \sum_{i=1}^{2l+1} f_i * \overline{C_{\tau,k,i}} \bar{H}_i$$

is identically zero implying that $g = 0$ which completes the proof of the theorem.

3. The reduced problem in $L_\infty(\mathbf{R}^N)$

Our main result depends on the following reduced problem in radial spectral analysis in $L_\infty(\mathbf{R}^N)$:

THEOREM 3.1. *The one-sided analogue of Wiener's Tauberian Theorem holds for G only if the following statement is true: For every non-trivial $f \in L_\infty(\mathbf{R}^N)$ there exist $R_0 \geqq 0$ and a non-trivial $\varphi \in L_\infty(\mathbf{R}^N)$ with $\mathrm{Sp}(\varphi) \subseteq S_{R_0}$ such that φ is contained in the w^*-closed subspace spanned by $\{f * \mu_r : r > 0\}$.*

PROOF. Suppose that the right-sided analogue of Wiener's Tauberian Theorem holds for G. By duality, this is equivalent to the fact that every w*-closed right-invariant subspace contains a minimal subspace.

Let $f \in L_\infty(\mathbf{R}^N)$ and let $\tilde{f} \in L_\infty(G)$ where $\tilde{f}(\sigma, x) = f(x)$. We may assume that $0 \in \mathrm{Sp}(f)$.

By Theorem 2.1, $M(\tilde{f})$ contains a function ψ of the form

$$\psi(\sigma, x) = \sum_i m_{i,j}(\sigma) f_i(x)$$

where $m_{i,j} \in \Gamma$ and $f_i \in L_\infty(\mathbf{R}^N)$ with $\mathrm{Sp}(f_i) \subseteq S_{R_0}$ for some $R_0 > 0$.

Let $\varphi \in L_1(\mathbf{R}^N)$ where $\int \varphi \bar{f}_1 dx \neq 0$. If $m_{i,j} \in \Gamma - \Gamma_0$ then the function $\psi_1 = m_{1,j} \otimes \varphi$ would be orthogonal to all right-translates of $\tilde{f}$.

But we have $\int_G \psi \bar{\Psi}_1 dg \neq 0$, a contradiction. Thus $m_{i,j} \in \Gamma_0$ and are the matricial coefficients of an irreducible representation of SO(N) based on the spherical harmonics $\{H_i\}$, $i = 1, 2, \ldots, 2l+1$. For some net $\{h_\tau\} \in L_1(\mathbf{R}^N)$ we have

$$\int_{\mathbf{R}^N} f(x - \sigma w) h_\tau(w) dw \xrightarrow[\tau]{w^*} \sum_{i=1}^{2l+1} m_{i,j}(\sigma) f_i(x)$$

implying that

$$\int_{\mathrm{SO}(N)} \left[\int_{\mathbf{R}^N} f(x - \xi) h_\tau(\sigma^{-1}\xi) d\xi \right] \bar{m}_{i,j}(\sigma) d\sigma \xrightarrow[\tau]{w^*} f_i(x)$$

for $i = 1, 2, \ldots, 2l+1$.

It follows that

$$\int_{\mathbf{R}^N} f(x - \xi) a_\tau(|\xi|) H_i\left(\frac{\xi}{|\xi|}\right) d\xi \xrightarrow[\tau]{w^*} f_i(x), \qquad i = 1, 2, \ldots, 2l+1,$$

where

$$a_\tau(|\xi|) = \int_{S_1} h_\tau(|\xi|\xi') H_j(\xi') d\xi'.$$

By taking Fourier transform we get

$$\hat{f}H_i\tilde{a}_\tau \xrightarrow[\tau]{w^*} \hat{f}_i \tag{3.1}$$

where $\widehat{a_\tau H_i} = \tilde{a}_\tau H_i$, $i = 1,2,\ldots,2l+1$.

Suppose now that $\hat{f}_1 \neq 0$. From (3.1) we obtain

$$\hat{f}_1 |H_i|^2 = \hat{f}_i H_1 \bar{H}_i, \qquad i = 1,2,\ldots,2l+1,$$

implying that

$$\sum_{i=1}^{2l+1} \hat{f}_i \bar{H}_i = \hat{f}_1 \left(\sum_{i=1}^{2l+1} |H_i|^2 \right) \neq 0.$$

Hence, by Theorem 2.2, $M(\psi) \subseteq M(\bar{f})$ contains a non-zero function $g(\sigma, x) = \varphi(x)$, $\varphi \in L_\infty(\mathbf{R}^N)$, $\mathrm{Sp}(\varphi) \subseteq S_{R_0}$.

That is, for some net $h_\tau \in L_1(\mathbf{R}^N)$ we have

$$\int f(x - \sigma w) h_\tau(w) dw \xrightarrow[\tau]{w^*} g(\sigma, x) = \varphi(x).$$

This implies that

$$\int f(x-\xi)\tilde{h}_\tau(|\xi|)d\xi \xrightarrow[\tau]{w^*} \varphi(x) \qquad \text{where } \tilde{h}_\tau(|\xi|) = \int_{\mathrm{SO}(N)} h_\tau(\sigma^{-1}\xi)d\sigma$$

which completes the proof of the theorem.

4. Pseudo-measures supported on spheres

Let $f \in L_\infty(\mathbf{R}^N)$ with $\mathrm{Supp}(\hat{f}) \subseteq S_1$ and let $\{H_{j,k}\}_{j=1}^{2k+1}$ be a basis of the spherical-harmonics of degree k. Then f has a "Fourier expansion" of the form

$$f(x) \sim \sum_{k=0}^{\infty} \sum_{j=1}^{2k+1} g_{j,k}(|x|) H_{j,k}\left(\frac{x}{|x|}\right)$$

where

$$g_{j,k}(|x|) = C_N \int_{|\xi|=|x|} f(\xi) \bar{H}_{j,k}\left(\frac{\xi}{|\xi|}\right) d\xi.$$

For each $\psi_{j,k} = g_{j,k}H_{j,k}$, $\mathrm{Supp}(\hat{\psi}_{j,k}) \subseteq S_1$ implying that

$$\hat{\psi}_{j,k} = D_{j,k}H_{j,k}$$

where $D_{j,k}$ are radial pseudo-measures supported on S_1. It follows by Varopoulos [6] that

$$D_{j,k} = \sum_{l=0}^{K} C_{j,kl}\mu_1^{(l)}$$

where

$$K = \left[\frac{N-1}{2}\right].$$

Hence, $\hat{f}$ admits a Fourier expansion of the form

$$\hat{f} \sim \sum_{k=0}^{\infty} \sum_{j=1}^{2l+1} \left[\sum_{l=0}^{K} C_{j,k,l}\mu_1^{(l)}\right] H_{j,k}.$$

To obtain estimates for $C_{j,k,l}$ we observe that

$$\psi_{j,k}(x) = \sum_{l=0}^{K} C_{j,k,l}\widehat{(\mu_1^{(l)}H_{j,k})}(x)$$

$$= \sum_{l=0}^{K} C_{j,k,l}P_{l,k}(|x|)H_{j,k}\left(\frac{x}{|x|}\right)$$

where

$$P_{l,k}(|x|) = i^l|x|^l\varphi_k(|x|) \quad \text{and} \quad \varphi_k \text{ is analytic in } z^2.$$

We have

$$\|\psi_{j,k}\|_\infty \leqq C_N\|f\|_\infty \qquad \text{for } k \geqq 0, \quad 1 \leqq j \leqq 2k+1. \tag{4.1}$$

Let $x_0 \in \mathbf{R}^N$, $x_0 \neq 0$ such that $\varphi_k(x_0) \neq 0$ and $H_{j,k}(x_0/|x_0|) \neq 0$ for $k \geqq 0$, $1 \leqq j \leqq 2k+1$. Our desired estimates follow now by (4.1) yielding

$$\left|\sum_{l=0}^{K} C_{j,k,l}i^l|x_0|^l\right| \leqq Q_k\|f\|_\infty, \qquad k \geqq 0, \quad 1 \leqq j \leqq 2k+1, \tag{4.2}$$

where Q_k does not depend on f.

5. The counter-example

The generalization of the main result in [8] to $N > 2$ is given by

THEOREM 5.1. *The one-sided analogue of Wiener's Tauberian Theorem fails to hold for $G = M(N)$. That is, there exists a proper closed right ideal in $L_1(G)$ which is not contained in any closed maximal right ideal.*

PROOF. Let T_h, $h>0$, denote the radial function on $\mathbf{R}^N$ defined as

$$T_h(x)=\begin{cases} h(|x|-1), & 1\leqq |x|<2,\\ h, & 2\leqq |x|<3,\\ h(4-|x|), & 3\leqq |x|<4,\\ 0, & \text{elsewhere.}\end{cases}$$

Let $b_k=\|K_k\|_1$ where $K_k\in L_1(\mathbf{R}^N)$ such that

$$\hat{K}_k(w)=H_{1,k}\left(\frac{w}{|w|}\right)\qquad \text{for } 2\leqq |w|\leqq 3,\quad k=0,1,\dots,$$

and let ψ be a radial function in $L_1(\mathbf{R}^N)$ with $\mathrm{Supp}(\hat{\psi})\subset\{w:2<|w|<3\}$ and $\|\psi\|_1=1$. For $\lambda\in\mathbf{R}$ let $T_h^{(\lambda)}$ denote the radial function defined by

$$T_h^{(\lambda)}(x)=e^{i\lambda|x|}T_h(|x|).$$

By a Riemann–Lebesgue type argument it follows that $\lim_{\lambda\to\infty}\|\widehat{T_h^{(\lambda)}}\|_\infty=0$ [8]. Let $\{a_k\}_{k=1}^\infty$ and $\{\lambda_k\}_{k=1}^\infty$ satisfy $a_k/Q_k\to\infty$ as $k\to\infty$ (Q_k are defined in (4.2)) and

$$\|T_{a_k}^{\lambda_k}\|_\infty\leqq 1/2^k b_k,\qquad k=0,1,\dots.$$

Let k_j be defined by

$$\hat{k}_0(x)=T_1(|x|)$$

$$\hat{k}_j(x)=T^{(\lambda_j)}(|x|),\qquad j=1,2,\dots.$$

Let $g_j=k_j*\psi$ and let $f\in L_\infty(\mathbf{R}^N)$ be given by

$$f(x)=\sum_{k=0}^\infty g_k*K_k(x)=\sum_{k=0}^\infty A_k^{(1)}(f;|x|)H_{1,k}\left(\frac{x}{|x|}\right).$$

Suppose now that some $\Phi\in L_\infty(\mathbf{R}^N)$ with $\mathrm{Sp}(\Phi)\subseteq S_{R_0}$, $R_0\geqq 0$, is contained in the w*-closed subspace spanned by $\{f*\mu_r:r\geqq 0\}$. Since $\mathrm{Sp}(f)\subseteq\mathrm{Supp}(\hat{\psi})$ we may assume that $2<R_0<3$. Let φ_τ be the net of radial functions in $L_1(\mathbf{R}^N)$ which satisfies

$$f*\varphi_\tau\xrightarrow[\tau]{w^*}\Phi.$$

We may assume that $\mathrm{Supp}\,\hat{\varphi}_\tau\subset\{x:2<|x|<3\}$. Let

$$\hat{\Phi}\sim\sum_{k=0}^\infty\sum_{j=1}^{2k+1}\left[\sum_{l=0}^K C_{j,k,l}\mu_{R_0}^{(l)}\right]H_{j,k}$$

be the Fourier expansion of Φ. One notices that $C_{j,k,l}=0$ for $j\neq 1$ and that

$$\hat{k}_n\hat{\psi}\hat{\varphi}_\tau \xrightarrow[\tau]{w^*} \sum_{l=0}^{K} C_{1,n,l}\mu_{R_0}^{(l)} \tag{5.1}$$

as pseudo-measures, $n=0,1,\dots$. It follows, by $n=0$, that

$$\hat{\psi}\hat{\varphi}_\tau \xrightarrow[\tau]{w^*} \sum_{l=0}^{K} C_{1,0,l}\mu_{R_0}^{(l)}$$

implying that

$$\hat{k}_n\sum_{l=0}^{K} C_{1,0,l}\mu_{R_0}^{(l)} = \sum_{l=0}^{K} C_{1,n,l}\mu_{R_0}^{(l)}.$$

By comparing the coefficients of $\mu_{R_0}^{(l)}$ we obtain

$$Z_m(\lambda_n)\hat{k}_n(R_0)=C_{1,n,m}$$

where

$$Z_m(x)=\sum_{l=m}^{K}(-1)^l\binom{l}{m}C_{1,0,l}(ix)^{l-m} \qquad \text{for } m=0,1,\dots,K.$$

From (4.2) we get

$$\left|\sum_{l=0}^{K}|x_0|^l Z_l(\lambda_n)\right|\,|\hat{k}_n(R_0)|\leqq Q_n\|\Phi\|_\infty \qquad \text{for } n=0,1,\dots.$$

Since

$$\frac{|\hat{k}_n(R_0)|}{Q_n}=\frac{a_n}{Q_n}\to\infty \quad \text{and} \quad \lambda_n\to\infty$$

it follows that $\sum_{l=0}^{K}|x_0|^l Z_l(x)\equiv 0$ implying that $C_{1,0,l}=0$ for $l=0,1,\dots,K$. Hence $\hat{\psi}\hat{\varphi}_\tau \xrightarrow[\tau]{w^*} 0$ implying, by 5.1, that $C_{1,n,l}=0$ for $n=1,2,\dots,\ l=0,1,\dots,K$.

It follows therefore that $\Phi=0$ which completes the proof of the theorem.

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